"Tables for the Solution of the Equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\cdot\frac{dy}{dx}-\left(1+\frac{n^2}{x^2}\right)y = 0."$$

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1. The object of the present paper is to exhibit the processes of calculation of the values of the two solutions of the equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\cdot\frac{dy}{dx}-\left(1+\frac{n^2}{x^2}\right)y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

for successive values of x in the two cases of $n = 0$ and $n = 1$.

That is if

$$y = \mathrm{AI}_n(x)+\mathrm{BK}_n(x)$$

be the complete integral of (1), where $\mathrm{K}_n(x)$ is a function which becomes zero when x is indefinitely increased, our object is to calculate the values of $\mathrm{I}_0(x)$, $\mathrm{I}_1(x)$, $\mathrm{K}_0(x)$, $\mathrm{K}_1(x)$ for successive equidistant values of x.

The values of $\mathrm{I}_0(x)$ and $\mathrm{I}_1(x)$ have been calculated and published by a committee of the British Association for the Advancement of Science. To the best of the writer's knowledge, no steps have been taken towards the computation of $\mathrm{K}_0(x)$ and $\mathrm{K}_1(x)$. The Tables I and II, at the end of this paper, give the values of these latter functions for intervals of 0·1 in the argument, to such a large number of decimal places as will make it a mere matter of difference calculation to determine intermediate values of $\mathrm{K}_0(x)$ and $\mathrm{K}_1(x)$ to any reasonable degree of accuracy, at any rate for values of x greater than unity; and also by means of the sequence laws to derive those of $\mathrm{K}_2(x)$, $\mathrm{K}_3(x)$ ......, as far as may be requisite.

It will be convenient to state a number of well-known theorems in regard to the solution of (1).

2. The function $\mathrm{I}_n(x)$ is defined by the condition

$$\mathrm{I}_n(x) = \sum_{r=0}^{r=\infty}\frac{\left(\frac{x}{2}\right)^{n+2r}}{\Pi(r)\,\Pi(n+r)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

For all values of n

$$y = \mathrm{AI}_n(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

is a solution of (1), $\Pi(n)$ being the function defined by Gauss.*

* 'Werke,' vol. 3, p. 145.

When n is a positive integer, a second solution is given by

$$y = \mathrm{B}\Lambda_n(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

where

$$\begin{aligned}\Lambda_n(x) = {} & \mathrm{I}_n(x)\log x \\ & + \frac{(-2)^{n-1}\Pi(n-1)}{x^n}\left\{1 - \frac{\left(\frac{x}{2}\right)^2}{1\,.\,\overline{n-1}} + \frac{\left(\frac{x}{2}\right)^4}{1\,.\,2\,.\,\overline{n-1}\,\overline{n-2}} - \ldots\ldots \right. \\ & \left. + \frac{(-1)^{n-1}\left(\frac{x}{2}\right)^{2n-2}}{\Pi(n-1)\,.\,\Pi(n-1)}\right\} \\ & - \tfrac{1}{2}\sum_{r=0}^{r=\infty}\frac{\left(\frac{x}{2}\right)^{n+2r}}{\Pi(n)\,\Pi(n+r)}(\mathrm{S}_r + \mathrm{S}_{n+r}) \quad \ldots\ldots\ldots\ldots \quad (5),\end{aligned}$$

where S_r denotes the series $1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{r}$. It is also necessary to assume zero as the value of the, in itself meaningless, symbol S_0.

It is also known that if E represent the quantity

$$\log 2 + \frac{\Gamma'(1)}{\Gamma(1)}$$

the function $\mathrm{EI}_n(x) - \Lambda_n(x)$ becomes indefinitely small as x increases indefinitely. Hence since in virtue of (3) and (4)

$$y = \mathrm{B}\{\mathrm{EI}_n(x) - \Lambda_n(x)\}$$

is obviously a solution of (1), it is allowable to write

$$\mathrm{K}_n(x) = \mathrm{EI}_n(x) - \Lambda_n(x) \quad \ldots\ldots\ldots\ldots \quad (6).$$

3. The three functions I, Λ, and K are all subject to the laws

$$\left.\begin{aligned}\frac{d}{dx}(x^{-n}\mathrm{I}_n) &= x^{-n}\mathrm{I}_{n+1} \\ \frac{d}{dx}(x^n\mathrm{I}_n) &= x^n\mathrm{I}_{n-1}\end{aligned}\right\}(\alpha),$$

where I_n is written for $\mathrm{I}_n(x)$. These equations hold when either Λ or K is substituted for I. When n has the value zero, the two equations must be replaced by the single equation

$$\frac{d\mathrm{I}_0}{dx} = \mathrm{I}_1,$$

or the same with Λ or K written for I.

These laws give, for values of n not less than unity,

$$\left.\begin{aligned} 2\frac{dI_n}{dx} &= I_{n+1}+I_{n-1} \\ \frac{2nI_n}{x} &= I_{n-1}-I_{n+1} \end{aligned}\right\} (\beta).$$

They are known and will be quoted as the sequence laws.

4. It can be shown that $K_n(x)$ is expressible in two ways in terms of a definite integral, namely,

$$K_n(x) = (-1)^n \frac{\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n \int_1^\infty \epsilon^{-px}(p^2-1)^{\frac{2n-1}{2}}dp \quad \ldots \quad (7),$$

$$K_n(x) = (-1)^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})}\cdot\left(\frac{2}{x}\right)^n \int_0^\infty \frac{\cos pxdp}{(1+p^2)^{\frac{2n+1}{2}}} \quad \ldots\ldots\ldots \quad (8).$$

By putting $p = 1+\frac{z}{x}$ in (7), expanding the binomial and integrating the separate terms, another form can be obtained for $K_n(x)$, namely,

$$K_n(x) = (-1)^n\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}\epsilon^{-x}\left\{1+\frac{4n^2-1}{8x}+\frac{(4n^2-1)(4n^2-9)}{1\cdot 2\,(8x)^2}+\ldots\ldots\right\} (9);$$

where the series within the bracket can be brought to a close at any point by means of a remainder term, which, after a certain point in the series, is always numerically less than the next term given by the general law of the series.

5. It is now possible to explain the processes by which the Tables I and II at the end of this paper have been calculated. The series actually employed are, for the smaller values of x, the ultimately convergent series (6); and for larger values the series (9).

In the calculation of the Λ functions, the natural logarithms of x are required. These the writer has taken from Wolfram's table at the end of Vega's 'Thesaurus Logarithmorum,' having, in the numbers up to 20 and for the prime numbers up to 59, verified them to 30 places of decimals by calculation.

The quantity E has been derived from Gauss.*

Using Gauss's notation

$$\psi(z) = \frac{\Pi'(z)}{\Pi(z)} = \frac{\Gamma'(z+1)}{\Gamma(z+1)}$$

it follows that $\quad E = \log 8+\psi(-\tfrac{1}{2}) = \log 2+\psi(0).$

* 'Werke,' vol. 3, p. 155.

From Wolfram's table, taking thirty-six places,

$$\log 2 = 0{\cdot}693\ 147\ 180\ 559\ 945\ 309\ 417\ 232\ 121\ 458\ 176\ 568.$$

The value of $\psi(0)$ is given in a note by Gauss as

$$\psi(0) = -0{\cdot}577\ 215\ 664\ 901\ 532\ 860\ 606\ 512\ 090\ 082\ 402\ 431.$$

The algebraical sum of these is

$$0{\cdot}115\ 931\ 515\ 658\ 412\ 448\ 810\ 720\ 031\ 375\ 774\ 137,$$

which is, therefore, the value of E to many more places than will be required.

The quantity $-\psi(0)$ is, of course, Euler's constant, and the above value is also to be derived from a paper by the late Professor J. C. Adams in the 'Proceedings of the Royal Society.'

6. The calculations of $I_0(x)$, $I_1(x)$, $K_0(x)$, $K_1(x)$ are best carried on in connection with one another. We have

$$K_0(x) = -I_0(x)\{\log x - E\} + \left\{\left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^4 \frac{S_2}{\{\Pi(2)\}^2} + \left(\frac{x}{2}\right)^6 \cdot \frac{S_3}{\{\Pi(3)\}^2} + \ldots\ldots\right\}.$$

$$K_1(x) = -I_1(x)\{\log x - E\} - \frac{1}{x} + \tfrac{1}{2}\left\{\left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^3 \frac{S_1 + S_2}{\Pi(1).\Pi(2)} + \ldots\ldots\right\}.$$

The first process is to find the values of $I_0(x)$ and $I_1(x)$.

If a series of quantities, β_0, β_1, $\beta_2, \ldots\ldots, \beta_{2r}, \beta_{2r+1}, \ldots\ldots$ be determined by the successive relations

$$\beta_{2r+1} = \frac{\frac{1}{2}x}{r+1} \cdot \beta_{2r}, \qquad \beta_{2r+2} = \frac{\frac{1}{2}x}{r+1} \cdot \beta_{2r+1},$$

coupled with the condition $\beta_0 = 1$, it is easily seen that

$$I_0(x) = \beta_0 + \beta_2 + \beta_4 + \ldots\ldots = \sum_{r=0}^{r=\infty} \beta_{2r},$$

$$I_1(x) = \beta_1 + \beta_3 + \beta_5 + \ldots\ldots = \sum_{r=0}^{r=\infty} \beta_{2r+1}.$$

Thus the successive terms of $I_0(x)$ and $I_1(x)$ are obtained by multiplying by a series of factors of the form $\frac{1}{2}x/r+1$; the alternate terms when obtained are written down underneath one another, the odd ones in one column, the even ones in another, and by addition of each column the values of $I_0(x)$ and $I_1(x)$ are obtained.

In working out the values of $I_0(x)$ and $I_1(x)$ given in Table I, all the

multiplications by $\frac{1}{2}x/r+1$ have been conducted in two different forms to avoid the possibility of mistakes. Thus, for instance, in working out $I_0(5{\cdot}2)$ and $I_1(5{\cdot}2)$, the factor $\frac{1}{2}x/8$, or $2{\cdot}6/8$, can be used as it stands, or as $\frac{13}{40}$, and also put into the form $\frac{1}{8}+\frac{1}{5}$. The adoption in all cases of two quite different processes is an almost infallible guide to the detection of a mistake.

7. The values of $I_0(x)$ and $I_1(x)$ being thus obtained, $K_0(x)$ and $K_1(x)$ can be derived.

We have

$$K_0(x) = -I_0(x)\{\log x - E\} + \left\{\left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^4 \frac{S_2}{\{\Pi(2)\}^2} + \left(\frac{x}{2}\right)^6 \frac{S_3}{\{\Pi(3)\}^2} + \dots\right\}$$

$$= -I_0(x)\{\log x - E\} + \{\beta_2 + \beta_4 S_2 + \beta_6 S_3 + \beta_8 S_4 + \dots\}.$$

But $0 = I_0(x) - 1 - \{\beta_2 + \beta_4 + \beta_6 + \beta_8 + \dots\dots\}$.

Adding

$$K_0(x) = -I_0(x)\{\log x - E - 1\} - 1 + \{\beta_4(S_2-1) + \beta_6(S_3-1) + \beta_8(S_4-1) + \dots\}.$$

It will be convenient to denote $\beta_{2r}(S_r - 1)$ by the symbol γ_{2r}. Hence

$$K_0(x) = -I_0(x)\{\log x - E - 1\} - 1 + \{\gamma_4 + \gamma_6 + \gamma_8 + \dots\dots\} \quad \dots\dots\dots \text{(10)}$$

The value of $I_0(x)$ is known and that of $\log x$ can be found from Wolfram's Table. The quantities γ_{2r} must be derived, each from the corresponding β_{2r}.

For earlier values of γ_{2r} the multiplier $S_r - 1$ is most easily used in the natural form

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\dots + \frac{1}{r}.$$

For later values, it is simpler to use the decimalized values of $S_r - 1$ given in Table V.

In using the primary form, many simplifications are possible, thus $\frac{1}{2}+\frac{1}{3} = \frac{5}{6} = \frac{10}{12}$, and the multiplication is effected by shifting the decimal point one place to the right, and dividing by 12.

Again,
$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4} = 1+\frac{1}{12},$$
$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} = 1+\frac{1}{12}+\frac{1}{5} = 1+\frac{1}{3}-\frac{1}{20};$$
$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6} = 1,$$
$$\frac{1}{4}+\frac{1}{5}+\frac{1}{10} = \frac{1}{2}+\frac{1}{20},$$
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{12} = 1,$$

and so on. In the computation of the values given in Table I, two

different processes for computing each γ have been employed, so that any mistake is almost certain to have been detected.

For the lower values of x a second process of calculating the values of γ_{2r} has been found from the obvious fact that if γ'_{2r} be the value of γ_{2r} when x becomes x/m,

$$\gamma'_{2r} = \frac{\gamma_{2r}}{m^{2r}}.$$

Thus the values of the quantities γ for $x = 2{\cdot}6$ can be deduced from those for $x = 5{\cdot}2$ by a series of divisions by 2 or powers of 2.

For the values of x from $x = 3{\cdot}1$ upwards, this process was not available, but either two different transformations of the sum of the vulgar fractions, or one such transformation, and the decimalized value have been used in every case.

Another process, which has been occasionally used, when the fraction $\frac{1}{2}x/r+1$ happened to be in low terms, is based on the easily proved formula

$$\gamma_{2r+2} = \left(\frac{\frac{x}{2}}{r+1}\right)^2 \cdot \gamma_{2r} + \frac{\beta_{2r+2}}{r+1}.$$

It only remains to multiply $I_0(x)$ by $(\log x - E - 1)$ and, adding unity to this product, to subtract the sum from $\Sigma\gamma$. The value of $K_0(x)$, which is always a positive quantity, is then obtained.

8. The second function $K_1(x)$ can be readily expressed in terms of quantities already found.

For

$$K_1(x) = -I_1(x)(\log x - E) - \frac{1}{x} + \tfrac{1}{2}\left\{\frac{x}{2} + \left(\frac{x}{2}\right)^3 \frac{S_1+S_2}{\Pi(1)\Pi(2)} + \left(\frac{x}{2}\right)^5 \frac{S_2+S_3}{\Pi(2)\Pi(3)} + \ldots\ldots\right\},$$

also

$$0 = I_1(x) - \left\{\frac{x}{2} + \left(\frac{x}{2}\right)^3 \frac{1}{\Pi(1)\Pi(2)} + \left(\frac{x}{2}\right)^5 \frac{1}{\Pi(1)\Pi(2)} + \ldots\ldots\right\},$$

adding

$$K_1(x) = -I_1(x)(\log x - E - 1) - \frac{1}{x} - \frac{x}{4}$$
$$+ \tfrac{1}{2}\left\{\left(\frac{x}{2}\right)^3 \frac{S_1+S_2-2}{\Pi(1)\Pi(2)} + \ldots\ldots + \left(\frac{x}{2}\right)^{2r+1} \frac{S_r+S_{r+1}-2}{\Pi(r)\Pi(r+1)} + \ldots\ldots\right\};$$

but $$\left(\frac{x}{2}\right)^3 \frac{S_1+S_2-2}{\Pi(1)\Pi(2)} = \tfrac{1}{2}\left(\frac{x}{2}\right)^3 \frac{1}{\Pi(1)\Pi(2)} = \frac{2}{x}\cdot\beta_4$$

$$\left(\frac{x}{2}\right)^5 \frac{S_2+S_3-2}{\Pi(2)\Pi(3)} = \left(\frac{x}{2}\right)^5 \cdot \frac{2(S_2-1)+\frac{1}{3}}{\Pi(2)\Pi(3)} = \frac{x}{3}\gamma_4 + \frac{2}{x}\beta_6$$

....

$$\left(\frac{x}{2}\right)^{2r+1}\frac{S_r+S_{r+1}-2}{\Pi(r)\Pi(r+1)} = \left(\frac{x}{2}\right)^{2r+1}\frac{2(S_r-1)+\dfrac{1}{r+1}}{\Pi(r)\Pi(r+1)}$$

$$= \frac{x}{r+1}\gamma_{2r}+\frac{2}{x}\beta_{2r+2}$$

....

Hence

$$\begin{aligned}
K_1(x) &= -I_1(x)(\log x - E - 1) - \frac{1}{x} - \frac{x}{4} \\
&\quad + \frac{1}{x}(\beta_4+\beta_6+\ldots\ldots+\beta_{2r+2}+\ldots\ldots) \\
&\quad + \frac{x}{2}\left\{\tfrac{1}{3}\gamma_4+\tfrac{1}{4}\gamma_6+\ldots\ldots+\frac{1}{r+1}\gamma_{2r}\ldots\ldots\right\} \\
&= -I_1(x)(\log x - E - 1) - \frac{1}{x} - \frac{x}{4} + \frac{1}{x}\{I_0(x) - 1 - \beta_2\} \\
&\quad + \frac{x}{2}\left\{\tfrac{1}{3}\gamma_4+\tfrac{1}{4}\gamma_6+\ldots\ldots+\frac{1}{r+1}\gamma_{2r}+\ldots\ldots\right\} \\
&= -I_1(x)(\log x - E - 1) - \frac{2}{x} - \frac{x}{2} + \frac{I_0(x)}{x} \\
&\quad + \frac{x}{2}\left\{\tfrac{1}{3}\gamma_4+\tfrac{1}{4}\gamma_6+\ldots\ldots+\frac{1}{r+1}\gamma_{2r}+\ldots\ldots\right\} \quad \ldots\ldots\ldots\ldots\ldots (11)
\end{aligned}$$

The calculation therefore only involves two operations which entail much labour, namely, the multiplication of $I_1(x)$ by $\{\log x - E - 1\}$ and the computation of the series.

$$\tfrac{1}{3}\gamma_4+\tfrac{1}{4}\gamma_6+\ldots\ldots+\frac{1}{r+1}\gamma_{2r}+\ldots\ldots$$

9. The quantities given in Table I have been computed by these formulæ. The multiplications of $I_0(x)$ and $I_1(x)$ by $(\log x - E - 1)$ have been worked, taking $(\log x - E - 1)$ as multiplicand, which saves a good deal of labour, as many of the lines of multiplication used in finding $K_0(x)$ occur again in finding $K_1(x)$. In all cases the multiplications have been carried to several places further than are used, or given, in the final results.

10. A process of verification has been applied to the values given in Table I, based on the following theorem.

It is easy to show that if $y = y_1$ and $y = y_2$ be any two different solutions of the fundamental equation (1),

$$y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx} = \frac{A}{x}.$$

In the particular case of $n = 0$, y_1 and y_2 may have the values $I_0(x$ and $K_0(x)$ respectively. Hence

$$I_0()\frac{dK_0(x)}{dx} - K_0(x)\frac{dI_0(x)}{dx} = \frac{A}{x},$$

or, by means of the sequence laws,

$$I_0(x)K_1(x) - K_0(x)I_1(x) = \frac{A}{x}.$$

Multiplying by x, and putting $x = 0$, it is easily seen that $A = -1$, consequently for all values of x,

$$I_0(x) \cdot K_1(x) - I_1(x) \cdot K_0(x) = -\frac{1}{x}.$$

The writer has to thank his friend Captain Makgill, R.E., of Waiuku, Auckland, for verifying by this formula most of the results obtained before the writer left New Zealand. For the others he has had to rely on his own verification.

The formula is an infallible indicator of any mistake in the values of β_s or γ_{2s}, or in the process of multiplication of $I_0(x)$ and $I_1(x)$ into $(\log x - E - 1)$. It obviously will not indicate an erroneous value of this last quantity. The values of $(\log x - E - 1)$ have been all calculated in two different ways, so as to avoid the possibility of mistake, but in order to give the greatest security, a table of the values employed is appended, and the writer hopes that if any mistake is detected, information of it may be sent to him, as it would be a very easy matter to supply the requisite correction to the values of $K_0(x)$ and $K_1(x)$.

As a final test of the accuracy of the results, the differences of the column for $K_0(x)$ have been calculated up to those of the seventeenth order. Up to this point they present in each set of differences a series of regularly decreasing quantities. In the differences of the eighteenth order this ceases to be the case with regard to the quantities at the lower end of the column. This is due to the accumulation of the effect of residual error in the last figures of the column of values of $K_0(x)$. The differences of the seventeenth order at the lower end of the column are quantities consisting of fifteen ciphers followed by six significant figures. Now since 2^{20} is greater than a million, it follows that a residual error of four-tenths of a unit in the last figure, in opposite directions in two consecutive values of $K_0(x)$ might possibly, after eighteen differentiations, produce an error of a unit in the sixth place from the end, consequently completely disorganise the sequence of the eighteenth differences which consist only of five figures. That this has actually happened in this case the writer has shown by examining the effect of adding to the values of $K_0(x)$ given in Table I the three additional

figures, two of them certainly correct, which he has calculated. The differences at the lower end of the table then become regular up to the twentieth order.

This process has not been applied to the $K_1(x)$ column, because the writer believes that, granted $K_0(x)$ correct, the verification formula above sufficiently proves the accuracy of $K_1(x)$. The values of the quantities in Table I are believed to be correct to the last figure given. A dot after the last figure indicates that it has been increased by unity, the first figure omitted being equal to or greater than 5.

11. Table II has been computed by means of the formula (9).

The remainder after s terms in the series involves the integral—

$$\int_0^\infty z^{n+s-\frac{1}{2}}\epsilon^{-z}\left(1+\frac{\theta z}{2x}\right)^{n-s-\frac{1}{2}}dz,$$

where θ is some proper fraction.

Now whatever n may be, after a time $n-s-\frac{1}{2}$ becomes negative. When s has reached such a value, inspection of (9) shows that the terms in the series thereafter are alternately positive and negative, inasmuch as a new negative factor is introduced in forming each successive coefficient. It is also evident that, from and after that point in the series, the quantity $\left(1+\frac{\theta z}{2x}\right)^{\frac{2n-2s-1}{2}}$ is numerically less than unity, and the remainder required at any point to give the value of $K_n(x)$ is numerically less than the next term in the series.

Consequently, after the alternation of signs has begun, the sums of s terms, $(s+1)$ terms, $(s+2)$ terms, &c., will be a series of quantities alternately greater and less than the value of $K_n(x)$. As long as the terms of the series diminish, it is possible in this way to obtain a set of quantities, continually approaching one another, between alternate pairs of which $K_n(x)$ must lie.

For the values $n = 0$, $n = 1$, (9) gives—

$$K_0x = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}}\epsilon^{-x}\left\{1-\frac{1}{8x}+\frac{1\cdot 9}{8\cdot 16x^2}-\frac{1\cdot 9\cdot 25}{8\cdot 16\cdot 24x^3}+\ldots\ldots\right\} \quad \ldots (12)$$

$$K_1(x) = -\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}\epsilon^{-x}\left\{1+\frac{3}{8x}-\frac{3\cdot 5}{8\cdot 16x^2}+\frac{3\cdot 5\cdot 21}{8\cdot 16\cdot 24x^3}-\ldots\ldots\right\} \quad \ldots (13)$$

12. In $K_0(x)$ the alternation of signs begins with the first term. Hence the sum of 1, 3, 5, ... terms is numerically greater than the value of $K_0(x)$, while the sum of 2, 4, 6, ... terms is less.

The $\overline{r+1}$th term is derived from the rth by multiplying by $(2r-1)^2/8rx$. As long as this factor is less than unity, the $\overline{r+1}$th term is less than the rth, and the terms continue to diminish. The $\overline{r+1}$th

term is least when r has the largest value, which makes $(2r-1)^2$ less than $8rx$. This gives $r = q$, where q is the integral part of

$$\tfrac{1}{2}\{2x+1+2(x^2+x)^{\frac{1}{2}}\}.$$

Hence the nearest approach of the limits, within which (12) confines the value of $K_0(x)$, is

$$= \left(\frac{\pi}{2x}\right)^{\frac{1}{2}}\epsilon^{-x}\frac{1\cdot9\cdot25\ldots(2q-1)^2}{\Pi(q)(8x)^q} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

It is evident that as $K_0(x)$ lies between the sum of q terms, and the sum of $q+1$ terms, the mean of these two sums is as near an approximation to the actual value of $K_0(x)$ as (12) will give. This mean cannot differ from $K_0(x)$ by quite half the quantity (14).

If x be an integer, the value of q is $2x$; thus, if $x = 1$ the third term is the smallest: when $x = 5$ the eleventh, when $x = 8$ the seventeenth, and so on. The limit of error, estimated by half the least term, is for $x = 1$, 0·0162; for $x = 2$, 0·0042; for $x = 5$, 0·000 000 022; and for larger values of x the limit becomes rapidly smaller.

For values of x as great as, or greater than, five, $K_0(x)$ can thus be determined with accuracy to seven or more places of decimals.

Very similar statements can be made with respect to the determination of $K_1(x)$ from (13).

13. From (12)

$$K_0(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}}\epsilon^{-x}\left\{1-\frac{1}{8x}+\frac{1\cdot9}{1\cdot2(8x)^2}\cdots\cdots\right\}$$

The multipliers, disregarding the sign, by which the coefficients of the powers of x within the bracket are derived, each from the preceding, are

$$\tfrac{1}{8},\ \tfrac{9}{16},\ \tfrac{25}{24},\ \tfrac{49}{32},\ \tfrac{81}{40},\ \tfrac{121}{48},\ \tfrac{169}{56},\ldots\ldots$$

Let these numbers be denoted by the symbols $m_1, m_2, m_3, \ldots\ldots$, and let $(\pi/2x)^{\frac{1}{2}}\epsilon^{-x}$ be called β_0. Then if a series of quantities $\beta_1, \beta_2, \beta_3, \ldots\ldots$, be derived by the successive relations

$$\beta_1 = m_1\beta_0x^{-1},\quad \beta_2 = m_2\beta_1x^{-1},\quad \beta_3 = m_3\beta_2x^{-1}, \ldots\ldots\ldots\ldots \quad (15)$$

it is evident that

$$\begin{aligned} K_0(x) &= \{\beta_0-\beta_1+\beta_2-\beta_3+\beta_4\ldots\ldots\} \\ &= (\beta_0+\beta_2+\beta_4+\ldots\ldots)-(\beta_1+\beta_3+\beta_5+\ldots\ldots) \end{aligned}$$

The relations (15) are adapted to logarithmic computation. For the value of β_0 two logarithms beside that of x are required. These are

$$\log\epsilon = 0\cdot434\ 2944\ 819;$$

$$\log\left(\frac{\pi}{2}\right)^{\frac{1}{2}} = 0\cdot098\ 0599\ 325.$$

With the help of these and the logarithm of x, that of β_0 can be easily ascertained, and then, if the logarithms of $m_1, m_2, m_3, \ldots\ldots$, be tabulated, it is easy to derive those of $\beta_1, \beta_2, \beta_3, \ldots\ldots$, in succession.

The logarithms of $m_1, m_2, \ldots\ldots$, as far as it has been necessary to use them in the construction of Table II, are given at the end of this paper in Table VI.

14. In going through the calculation, it is, of course, useless to take the values of the quantities $\beta_1, \beta_2, \ldots\ldots$, to a decimal place further than the last one which can be accurately obtained in β_0. If ten-figure logarithms be used, ten significant figures can be ordinarily obtained with accuracy from the logarithm. Of this the writer has satisfied himself by working out the value of $(\pi/2x)^{\frac{1}{2}}\epsilon^{-x}$ by elementary arithmetic and the exponential theorem, for one or two simple values of x, as $x = 8$, $x = 11$, and comparing the result so obtained with that derived from the logarithms. They always agree for ten places, sometimes for eleven, if account be taken of the second differences of the logarithms.

It follows that for larger values of x, for which the smallest term in the series is less than $10^{-10}\beta_0$, the value of $\mathrm{K}_0(x)$ can be obtained with accuracy, probably for ten, and pretty certainly for nine significant figures. The tenth figure may be in error owing to the accumulation, in addition, of the errors in the last places of the quantities $\beta_1, \beta_2, \ldots\ldots$

15. Equation (13) gives

$$\mathrm{K}_1(x) = -\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}\epsilon^{-x}\left\{1+\frac{3}{8x}-\frac{3\cdot 5}{1\cdot 2(8x)^2}+\ldots\ldots\right\}.$$

The multipliers, disregarding sign, by which the coefficients of the successive powers of x within the bracket are derived, each from the preceding, are

$$\tfrac{3}{8},\ \tfrac{5}{16},\ \tfrac{21}{24},\ \tfrac{45}{32},\ldots\ldots$$

Let these be denoted by the symbols $\mu_1, \mu_2, \mu_3, \ldots\ldots$, and let a series of quantities $\beta'_1, \beta'_2, \beta'_3, \ldots\ldots$, be obtained from β_0 by the successive relations

$$\beta'_1 = \mu_1\beta_0 x^{-1}, \quad \beta'_2 = \mu_2\beta'_1 x^{-1}, \quad \beta'_3 = \mu_3\beta_2 x^{-1}, \ldots\ldots\ldots \quad (16)$$

β_0 having the same value as in Article (13).

Then evidently

$$\begin{aligned}\mathrm{K}_1(x) &= -\{\beta_0+\beta'_1-\beta'_2+\beta'_3-\beta'_4+\ldots\ldots\}\\ &= -[(\beta_0+\beta'_1+\beta'_3+\ldots\ldots)-(\beta'_2+\beta'_4+\ldots\ldots)];\end{aligned}$$

the summations being carried on, either until the smallest term of the series is reached, in the case of the lower values of x, or until a term is arrived at which is less than $10^{-10}\beta_0$, which will happen first for larger values of x.

The relations (16) are adapted to logarithmic computation. The logarithms of $\mu_1, \mu_2, \mu_3, \ldots$ are given in Table VI.

16. The verification of the values of $K_0(x)$, $K_1(x)$ in Table II, cannot be conducted on the method applied to those in Table I, because the values of $I_0(x)$, $I_1(x)$ are wanting.

A certain amount of check is given by the values of the four functions I_0, I_1, K_0, K_1, calculated for the integral values of x, by the former method, given in Table III.

Two other checks, in addition to the useful one of performing all additions and multiplications in two different ways, have been applied throughout.

The first depends on a very simple relation between the quantities β_r and β'_r.

It is easily seen, from the general formula for the $\overline{r+1}$th term in (9), that

$$\frac{\beta'_r}{\beta_r} = \frac{3\cdot5\cdot21\ldots\ldots\{(2r-1)^2-4\}}{1\cdot3^2\cdot5^2\ldots\ldots\quad(2r-1)^2};$$

which, since $(2r-1)^2-4=(2r+1)(2r-3)$, easily reduces to

$$(2r+1)/(2r-1).$$

Thus $$\beta'_r = \frac{2r+1}{2r-1}\beta_r = \beta_r\left(1+\frac{2}{2r-1}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

When the quantities β and β' have been calculated from the logarithmic formulæ, this result gives an easy method of verification. It detects any mistake in the computation of the logarithms, or in the derivation of the number from the logarithm.

This formula leaves untouched the possibility of a mistake in the value of β_0. To check this another process has been used.

17. If $f(x)$ be any continuous function of x, whose differential coefficients are also finite and continuous for the values of x considered, Taylor's Theorem gives

$$f(x+h) = f(x) + hf^{\text{i}}(x) + \frac{h^2}{\Pi(2)}f^{\text{ii}}(x) + \ldots\ldots$$

Let u_0 be the value of $f(x)$ corresponding to any particular value x_0 of x, and let $u_1, u_2, u_3, \ldots$ denote the values of $f(x_0+h)$, $f(x_0+2h)$, $f(x_0+3h), \ldots$ Similarly, let $u_{-1}, u_{-2}, u_{-3} \ldots$ denote $f(x_0-h)$, $f(x_0-2h)$, $f(x_0-3h), \ldots$

Then $$\frac{u_1-u_{-1}}{2h} = f^{\text{i}}(x_0) + \frac{h^2}{6}f^{\text{iii}}(x_0) + \frac{h^4}{120}f^{\text{v}}(x_0) + \ldots\ldots$$

If h be so small that the terms of the series on the right hand not written down may be neglected, and the three terms written down be denoted by u, v, w, respectively, it follows that

$$u+v+w = \frac{u_1-u_{-1}}{2h} = \alpha \text{ say};$$

writing $2h$ for h, this gives

$$u+4v+16w = \frac{u_2-u_{-2}}{4h} = \beta,$$

and putting $3h$ for h

$$u+9v+81w = \frac{u_3-u_{-3}}{6h} = \gamma.$$

From these three equations u can be found in terms of α, β, γ, and its value can be put into the convenient shape

$$u = \alpha-\tfrac{1}{2}(\beta-\alpha)+\tfrac{1}{10}(\gamma-\beta) \qquad \text{.....................(18)}$$

From the sequence law it follows that

$$K_1 = dK_0/dx.$$

Consequently, if values of $K_0(x)$ for seven equidistant values of x be taken, the quantities α, β, and γ can be derived, and (18) ought to give a value of u equal to that of $K_1(x)$ for the middle value of x. This test has been freely applied throughout Table II with very satisfactory results.

18. If the values of $f(x_0+4h)$ and $f(x_0-4h)$ be taken into account, a still more stringent test is afforded. As before, let these quantities be denoted by u_4 and u_{-4}. Let z be used to denote

$$\frac{h^6}{\Pi(7)}f^{\text{vii}}(x_0) \text{ and let } \frac{u_4-u_{-4}}{8h} = \delta.$$

Then

$$\begin{aligned} u+v+w+z &= \alpha, \\ u+4v+16w+256z &= \beta, \\ u+9v+81w+729z &= \gamma, \\ u+16v+256w+4096z &= \delta; \end{aligned}$$

whence it is not difficult to show that

$$u = \alpha-\tfrac{3}{5}(\beta-\alpha)+\frac{\gamma-\beta}{5}-\frac{\delta-\gamma}{35} \qquad \text{.................. (19)}$$

This can be used independently, or it can be made to yield a correction to (18). In the latter case the quantity

$$\tfrac{1}{10}(\beta-\alpha)+\tfrac{1}{35}(\delta-\gamma)-\tfrac{1}{10}(\gamma-\beta) \qquad \text{.............. (20)}$$

has to be subtracted from the value of u given by (18).

19. As an example, suppose the value of x_0 is taken as $7{\cdot}4$. Table II gives

$$
\begin{aligned}
u_{-4} &= 0{\cdot}000\ 424\ 795\ 741 \\
u_{-3} &= 0{\cdot}000\ 381\ 739\ 385 \\
u_{-2} &= 0{\cdot}000\ 343\ 079\ 156 \\
u_{-1} &= 0{\cdot}000\ 308\ 362\ 213 \\
u_{1} &= 0{\cdot}000\ 249\ 177\ 616 \\
u_{2} &= 0{\cdot}000\ 224\ 020\ 677 \\
u_{3} &= 0{\cdot}000\ 201\ 420\ 050 \\
u_{4} &= 0{\cdot}000\ 181\ 113\ 953
\end{aligned}
$$

whence, remembering that $h = \frac{1}{10}$, it easily follows that, a minus sign being understood before all the numbers,

$$
\begin{aligned}
\alpha &= 0{\cdot}000\ 295\ 922\ 985 \\
\beta &= 0{\cdot}000\ 297\ 646\ 197 \\
\gamma &= 0{\cdot}000\ 300\ 532\ 225 \\
\delta &= 0{\cdot}000\ 304\ 602\ 235
\end{aligned}
$$

whence

$$
\begin{aligned}
\beta-\alpha &= 0{\cdot}000\ 001\ 723\ 212 \\
\gamma-\beta &= 0{\cdot}000\ 002\ 886\ 028 \\
\delta-\gamma &= 0{\cdot}000\ 004\ 070\ 010
\end{aligned}
$$

Hence, using the formula (18),

$$
\begin{aligned}
\alpha &= 0{\cdot}000\ 295\ 922\ 985 \\
\tfrac{1}{10}(\gamma-\beta) &= 0{\cdot}000\ 000\ 288\ 603 \\
\hline
&\ 0{\cdot}000\ 296\ 211\ 588 \\
\tfrac{1}{2}(\beta-\alpha) &= 0{\cdot}000\ 000\ 861\ 606 \\
\hline
&\ 0{\cdot}000\ 295\ 349\ 982
\end{aligned}
$$

The value in the table for $K_1\,(7{\cdot}4)$ is $0{\cdot}000\ 295\ 349\ 978$.

If we apply the correction (20), the above values give

$$
\begin{aligned}
\tfrac{1}{10}(\beta-\alpha) &= 000\ 000\ 172\ 321 \\
\tfrac{1}{35}(\delta-\gamma) &= 116\ 286 \\
\hline
&\ 288\ 607 \\
\tfrac{1}{10}(\gamma-\beta) &= 288\ 603 \\
\hline
\text{Correction} &= 000\ 000\ 000\ 004
\end{aligned}
$$

This has to be subtracted from the former value, and the result agrees exactly with the value of $K_1(7{\cdot}4)$ in Table II.

The agreement is not in all cases quite so exact as in this example, as may be expected from the necessary existence of more or less of error in the last figures taken into account.

A slight additional verification of the general accuracy of Table II has been gained by the calculation of the term β_0 for the values 8, 9, 10, 11, and 12 by elementary arithmetic and the exponential theorem without the use of logarithms.

The last figure of the quantities in Table II cannot be depended on for strict accuracy, in which respect the table differs from Table I.

20. A farther extension of the formulæ of Articles 17 and 18 has some interest.

If, with the same notation extended, the quantity $(u_5 - u_{-5})/10h$ be denoted by ϵ, it is not difficult to prove that

$$u = \alpha - \tfrac{2}{3}(\beta - \alpha) + \tfrac{2}{7}(\gamma - \beta) - \frac{\delta - \gamma}{14} + \frac{\epsilon - \delta}{126} \,\cdots\cdots\cdots \quad (21).$$

This value of u can most easily be computed by subtracting

$$\tfrac{1}{6}(\beta - \alpha) + \tfrac{1}{70}(\gamma - \beta) + \tfrac{1}{14}(\delta - \gamma) - \tfrac{1}{5}(\gamma - \beta) - \frac{\epsilon - \delta}{126}$$

from the value of u given in (18).

This farther correction is too small to be applied with any certainty to the values of $K_1(x)$ derived from $K_0(x)$ in Table II. Obviously however, all these formulæ may be equally well applied to Table I, and throughout the range of that table, this formula deduces a value of $K_1(x)$ more accurate to one or two places than that given in (19).

To give two examples; one from the earlier part of the table.

If $x = 2{\cdot}6$

Equation (18) gives	$-K_1(x)$	$=$	0·065 284 052 521 550
" (19) "	$-K_1(x)$	$=$	0·065 284 044 927 362
" (21) "	$-K_1(x)$	$=$	0·065 284 045 062 511
while the correct value is			0·065 284 045 058 531

Again taking the largest value of x in Table I which admits of the application of (21), namely $x = 5{\cdot}5$,

Equation (18) gives	$-K_1(x)$	$=$	0·002 325 569 051 888
" (19) "	$-K_1(x)$	$=$	0·002 325 569 008 660
" (21) "	$-K_1(x)$	$=$	0·002 325 569 008 850
while the correct value is			0·002 325 569 008 849 005

None of these formulæ is sufficient for verification of the values in Table I to the last figure given.

Table I.

x.	$I_0(x)$.	$I_1(x)$.
0·1	1·002 501 562 934 095 601 400	0·050 062 526 047 092 692 114·
0·2	1·010 025 027 795 145 835 263·	0·100 500 834 028 125 115 768
0·3	1·022 626 879 351 596 991 120·	0·151 693 840 003 592 780 329·
0·4	1·040 401 782 229 341 241 022·	0·204 026 755 733 570 596 281
0·5	1·063 483 370 741 323 519 263	0·257 894 305 390 896 316 362
0·6	1·092 045 364 317 339 541 841·	0·313 704 025 604 922 130 966
0·7	1·126 303 018 306 809 198 051·	0·371 879 677 777 008 654 743·
0·8	1·166 514 922 869 802 731 431	0·432 864 802 620 639 821 166·
0·9	1·212 985 165 728 684 317 724·	0·497 126 448 160 964 276 677
1·0	1·266 065 877 752 008 335 598	0·565 159 103 992 485 027 208·
1·1	1·326 160 183 712 652 485 589	0·637 488 876 453 881 892 572·
1·2	1·393 725 584 134 064 395 588	0·714 677 941 552 643 086 231
1·3	1·469 277 797 944 250 888 664	0·797 329 314 979 268 902 964
1·4	1·553 395 099 731 216 509 982·	0·886 091 981 414 327 353 583·
1·5	1·646 723 189 772 890 844 876	0·981 666 428 577 907 585 652
1·6	1·749 980 639 738 909 390 905	1·084 810 635 129 879 617 220·
1·7	1·863 964 962 073 839 671 192	1·196 346 565 634 482 268 430·
1·8	1·989 559 356 618 050 914 345	1·317 167 230 391 898 987 579·
1·9	2·127 740 194 053 887 856 891	1·448 244 373 054 888 953 884·
2·0	2·279 585 302 336 067 267 437	1·590 636 854 637 329 063 382
2·1	2·446 283 129 436 182 291 275	1·745 499 808 836 106 159 137
2·2	2·629 142 863 567 314 172 737·	1·914 094 650 586 386 159 283
2·3	2·829 605 600 627 585 665 907	2·097 800 027 517 421 476 844·
2·4	3·049 256 657 989 413 844 196·	2·298 123 812 543 222 324 570
2·5	3·289 839 144 050 123 035 706·	2·516 716 245 288 698 441 528
2·6	3·553 268 904 243 671 659 925·	2·755 384 340 504 706 456 568
2·7	3·841 650 976 595 934 202 977	3·016 107 693 161 405 855 985
2·8	4·157 297 703 500 820 202 310	3·301 055 822 635 087 581 928
2·9	4·502 748 661 326 274 366 311·	3·612 607 212 436 907 736 703·
3·0	4·880 792 585 865 024 085 611	3·953 370 217 402 609 396 479·
3·1	5·294 491 489 675 606 473 324·	4·326 206 027 313 598 387 154
3·2	5·747 207 187 180 549 677 026·	4·734 253 894 709 620 419 983·
3·3	6·242 630 465 183 028 963 790	5·180 958 855 355 928 605 292
3·4	6·784 813 160 431 586 596 268·	5·670 102 192 635 219 559 794
3·5	7·378 203 432 225 479 660 344·	6·205 834 922 258 365 473 623·
3·6	8·027 684 547 054 009 945 933	6·792 714 601 361 299 242 400
3·7	8·738 617 524 169 395 584 970	7·435 745 796 535 335 730 518·
3·8	9·516 888 026 098 957 047 396	8·140 424 578 907 955 806 110
3·9	10·368 957 916 732 943 985 764	8·912 787 451 362 725 689 348·
4·0	11·301 921 952 136 330 496 356	9·759 465 153 704 449 909 475
4·1	12·323 570 116 019 571 436 934·	10·687 741 836 417 761 231 468·
4·2	13·442 456 163 297 646 200 379	11·705 620 143 051 615 977 998·
4·3	14·667 972 991 845 562 465 006	12·821 892 795 648 573 301 862
4·4	16·010 435 524 946 996 723 558·	14·046 221 337 533 105 734 577·
4·5	17·481 171 855 609 276 043 133	15·389 222 753 735 923 892 694·
4·6	19·092 623 479 519 459 002 267	16·862 564 761 976 656 391 871
4·7	20·858 455 526 644 462 400 770·	18·479 070 647 133 100 245 291
4·8	22·793 677 993 105 797 960 124	20·252 834 600 238 559 989 488·
4·9	24·914 779 075 837 756 060 699	22·199 348 620 092 491 190 354·
5·0	27·239 871 823 604 446 894 544	24·335 642 142 450 527 199 143
5·1	29·788 855 440 238 848 499 153	26·680 435 679 477 119 089 197
5·2	32·583 592 710 613 699 532 308	29·254 309 881 798 348 760 365
5·3	35·648 105 168 113 101 763 145	32·079 891 578 297 025 753 268
5·4	39·008 787 785 625 836 242 827	35·182 058 506 083 583 786 328·
5·5	42·694 645 151 847 784 559 282	38·588 164 616 327 393 255 945·
5·6	46·737 551 292 637 286 856 629	42·328 288 032 466 848 420 202·
5·7	51·172 535 515 159 998 128 205	46·435 503 947 521 351 864 819
5·8	56·038 096 892 622 866 750 874	50·946 184 978 774 806 273 857
5·9	61·376 550 271 771 251 908 395	55·900 331 753 160 078 871 856·
6·0	67·234 406 976 477 975 326 188	61·341 936 777 640 237 861 329

Table I.

$K_0(x)$.	$-K_1(x)$.	x.
2·427 069 024 702 016 612 519·	9·853 844 780 870 606 134 849·	0·1
1·752 703 855 528 145 906 617	4·775 972 543 220 472 248 750·	0·2
1·372 460 060 544 297 376 645·	3·055 992 033 457 324 978 851·	0·3
1·114 529 134 524 434 406 170·	2·184 354 424 732 687 379 723	0·4
0·924 419 071 227 665 861 782·	1·656 441 120 003 300 893 696	**0·5**
0·777 522 091 904 729 289 468·	1·302 834 939 763 502 176 671	0·6
0·660 519 859 915 101 548 740	1·050 283 535 312 917 951 430	0·7
0·565 347 105 265 895 668 369	0·861 781 634 472 180 346 690·	0·8
0·486 730 308 162 900 521 582·	0·716 533 578 776 019 074 786	0·9
0·421 024 438 240 708 333 336·	0·601 907 230 197 234 574 738·	**1·0**
0·365 602 391 543 185 880 566·	0·509 760 027 167 027 048 822·	1·1
0·318 508 220 286 593 615 118·	0·434 592 391 060 715 038 502·	1·2
0·278 247 646 300 026 999 011	0·372 547 495 631 962 166 173·	1·3
0·243 655 061 181 541 893 927·	0·320 835 902 229 875 750 946·	1·4
0·213 805 562 647 525 736 722·	0·277 387 800 456 843 816 085	**1·5**
0·187 954 751 969 332 325 059	0·240 633 911 357 611 855 164·	1·6
0·165 496 318 056 996 539 364·	0·209 362 488 204 082 474 675·	1·7
0·145 931 400 489 827 981 234	0·182 623 099 801 746 979 604·	1·8
0·128 845 979 276 047 479 856·	0·159 660 153 032 667 610 382	1·9
0·113 893 872 749 533 435 653·	0·139 865 881 816 522 427 285·	**2·0**
0·100 783 740 889 966 945 812·	0·122 746 411 533 507 910 608·	2·1
0·089 269 005 671 601 745 130·	0·107 896 810 119 087 275 030·	2·2
0·079 139 933 002 093 626 828	0·094 982 443 845 362 636 833	2·3
0·070 217 341 543 415 895 531	0·083 724 838 754 832 182 453·	2·4
0·062 347 553 200 366 186 029'	0·073 890 816 347 747 063 649·	**2·5**
0·055 398 303 286 321 951 484	0·065 284 045 058 531 495 000	2·6
0·049 255 400 915 817 592 455	0·057 738 398 956 525 947 419·	2·7
0·043 819 981 975 498 528 903	0·051 112 685 607 272 438 995	2·8
0·039 006 234 566 223 424 101·	0·045 286 423 298 361 443 561	2·9
0·034 739 504 386 279 248 072	0·040 156 431 128 194 184 377·	**3·0**
0·030 954 708 038 041 442 502·	0·035 634 054 949 617 493 670	3·1
0·027 594 997 675 100 610 315	0·031 642 895 211 398 770 897	3·2
0·024 610 632 145 839 314 335·	0·028 116 934 272 716 612 255·	3·3
0·021 958 018 806 808 280 394·	0·024 998 984 123 186 272 784	3·4
0·019 598 897 170 368 489 108·	0·022 239 392 925 923 833 739·	**3·5**
0·017 499 641 018 146 603 343	0·019 794 962 019 720 617 134·	3·6
0·015 630 659 921 626 661 612	0·017 628 035 102 223 266 688·	3·7
0 013 965 884 534 245 617 659·	0·015 705 729 078 473 492 808·	3·8
0·012 482 322 757 249 775 684	0·013 999 282 082 274 828 044·	3·9
0·011 159 676 085 853 024 270·	0·012 483 498 887 268 431 470	**4·0**
0·009 980 007 227 840 242 646·	0 011 136 277 633 479 931 554	4·1
0·008 927 451 541 542 371 598	0·009 938 204 735 917 087 547·	4·2
0·007 987 966 031 764 522 372	0·008 872 207 188 591 397 612·	4·3
0·007 149 110 623 307 253 932·	0·007 923 253 361 445 598 749·	4·4
0·006 399 857 243 233 975 046·	0·007 078 094 908 968 089 693·	**4·5**
0·005 730 422 917 292 834 887	0·006 325 043 644 264 015 020·	4·6
0·005 132 123 648 454 615 086·	0·005 653 778 240 030 826 704	4·7
0·004 597 246 316 724 657 899	0·005 055 176 444 056 299 816·	4·8
0·004 118 936 235 515 888 790·	0 004 521 169 177 299 838 509	4·9
0·003 691 098 334 042 594 275·	0·004 044 613 445 452 164 208	**5·0**
0·003 308 310 218 017 464 327·	0·003 619 181 462 317 798 328·	5·1
0·002 965 745 601 029 581 462	0·003 239 263 773 089 456 376	5·2
0·002 659 106 803 389 557 342·	0·002 899 884 491 690 688 906	5·3
0·002 384 565 189 724 900 197	0·002 596 627 040 177 797 776·	5·4
0·002 138 708 565 950 287 432·	0·002 325 569 008 849 005 155	**5·5**
0·001 918 494 684 356 577 228	0·002 083 224 950 609 789 166	5·6
0·001 721 210 115 723 315 288	0·001 866 496 088 311 830 924	5·7
0·001 544 433 842 281 102 204·	0·001 672 626 054 141 651 512	5·8
0·001 386 005 007 304 947 106·	0·001 499 161 899 722 485 306·	5·9
0·001 243 994 328 013 123 085	0·001 343 919 717 735 509 006·	**6·0**

Table II.

x.	$K_0(x)$.	$-K_1(x)$.	x.
5·0	0·003 691 098	0·004 044 614	5·0
5·1	0·003 308 310	0·003 619 182	5·1
5·2	0·002 965 746	0·003 239 264	5·2
5·3	0·002 659 107	0·002 899 884	5·3
5·4	0·002 384 565	0·002 596 627	5·4
5·5	0·002 138 709	0·002 325 569	5·5
5·6	0·001 918 495	0·002 083 225	5·6
5·7	0·001 721 210	0·001 866 496	5·7
5·8	0·001 544 433 7	0·001 672 626 1	5·8
5·9	0·001 386 005 0	0·001 499 161 9	5·9
6·0	0·001 243 994 3	0·001 343 919 7	6·0
6·1	0·001 116 678 7	0·001 204 954 3	6·1
6·2	0·001 002 518 9	0·001 080 532 4	6·2
6·3	0·000 900 139 2	0·000 969 108 8	6·3
6·4	0·000 808 309 9	0·000 869 305 8	6·4
6·5	0·000 725 931 8	0·000 779 894 4	6·5
6·6	0·000 652 021 37	0·000 699 777 68	6·6
6·7	0·000 585 699 16	0·000 627 976 68	6·7
6·8	0·000 526 178 09	0·000 563 617 16	6·8
6·9	0·000 472 753 79	0·000 505 918 31	6·9
7·0	0·000 424 795 74	0·000 454 182 49	7·0
7·1	0·000 381 739 385	0·000 407 786 222	7·1
7·2	0·000 343 079 156	0·000 366 172 174	7·2
7·3	0·000 308 362 213	0·000 328 841 997	7·3
7·4	0·000 277 182 870	0·000 295 349 978	7·4
7·5	0·000 249 177 617	0·000 265 297 390	7·5
7·6	0·000 224 020 678	0·000 238 327 458	7·6
7·7	0·000 201 420 050	0·000 214 120 873	7·7
7·8	0·000 181 113 953	0·000 192 391 797	7·8
7·9	0·000 162 867 668	0·000 172 884 307	7·9
8·0	0·000 146 470 705 2	0·000 155 369 211 8	8·0
8·1	0·000 131 734 278 6	0·000 139 641 228 9	8·1
8·2	0·000 118 489 040 5	0·000 125 516 451 2	8·2
8·3	0·000 106 583 050 1	0·000 112 830 094 0	8·3
8·4	0·000 095 880 013 8	0·000 101 434 481 3	8·4
8·5	0·000 086 257 566 3	0·000 091 197 247 7	8·5

x.	$K_0(x)$.	$-K_1(x)$.	x.
8·5	0·000 086 257 566 3	0·000 091 197 247 7	8·5
8·6	0·000 077 605 920 7	0·000 081 999 731 8	8·6
8·7	0·000 069 826 521 36	0·000 073 735 540 60	8·7
8·8	0·000 062 830 892 86	0·000 066 309 267 33	8·8
8·9	0·000 056 539 599 34	0·000 059 635 344 08	8·9
9·0	0·000 050 881 312 956	0·000 053 637 016 382	9·0
9·1	0·000 045 791 979 331	0·000 048 245 426 023	9·1
9·2	0·000 041 214 069 631	0·000 043 398 790 454	9·2
9·3	0·000 037 095 910 423	0·000 039 041 668 525	9·3
9·4	0·000 033 391 083 017	0·000 035 124 303 368	9·4
9·5	0·000 030 057 884 958	0·000 031 602 034 110	9·5
9·6	0·000 027 058 847 266	0·000 028 434 769 224	9·6
9·7	0·000 024 360 301 507	0·000 025 586 514 844	9·7
9·8	0·000 021 931 991 556	0·000 023 024 952 359	9·8
9·9	0·000 019 746 725 314	0·000 020 721 059 930	9·9
10·0	0·000 017 780 062 316	0·000 018 648 773 453 9	10·0
10·1	0·000 016 010 033 412	0·000 016 784 682 675	10·1
10·2	0·000 014 416 889 253	0·000 015 107 758 866	10·2
10·3	0·000 012 982 874 576	0·000 013 599 110 702	10·3
10·4	0·000 011 692 025 596	0·000 012 241 765 367	10·4
10·5	0·000 010 529 988 143	0·000 011 020 472 310	10·5
10·6	0·000 009 483 854 408	0·000 009 921 527 234	10·6
10·7	0·000 008 542 016 344 7	0·000 008 932 614 226	10·7
10·8	0·000 007 694 034 041 2	0·000 008 042 664 131 7	10·8
10·9	0·000 006 930 517 517 5	0·000 007 241 727 523 8	10·9
11·0	0·000 006 243 020 547 6	0·000 006 520 860 674 6	11·0
11·1	0·000 005 623 945 302 6	0·000 005 872 023 261 0	11·1
11·2	0·000 005 066 456 681 9	0·000 005 287 986 539 5	11·2
11·3	0·000 004 564 405 350 1	0·000 004 762 250 929 6	11·3
11·4	0·000 004 112 258 592 2	0·000 004 288 972 021 7	11·4
11·5	0·000 003 705 038 165 4	0·000 003 862 894 145 3	11·5
11·6	0·000 003 338 264 475 1	0·000 003 479 290 732 4	11·6
11·7	0·000 003 007 906 380 0	0·000 003 133 910 741 2	11·7
11·8	0·000 002 710 336 093 0	0·000 002 822 930 559 3	11·8
11·9	0·000 002 442 288 637 0	0·000 002 542 910 795 2	11·9
12·0	0·000 002 200 825 397 302	0·000 002 290 757 464 767	12·0

Table III.

x.	$I_0(x)$.	$I_1(x)$.	$K_0(x)$.	$-K_1(x)$.	x.
6·0	67·234 406 976 477 975 326	61·341 936 777 640 237 861	0·001 243 994 328 013 123	0·001 343 919 717 735 509	6·0
7·0	168·593 908 510 289 698 857	156·039 092 869 955 453 462	0·000 424 795 741 869 231	0·000 454 182 486 884 898	7·0
8·0	427·564 115 721 804 785 175	399·873 136 782 560 098 228	0·000 146 470 705 222 804	0·000 155 369 211 804 984	8·0
9·0	1093·588 354 511 374 695 845	1030·914 722 516 956 444 428	0·000 050 881 312 956 458	0·000 053 637 016 379 453	9·0
10·0	2815·716 628 466 254 471 294	2670·988 303 701 254 654 247	0·000 017 780 062 316 066	0·000 018 648 773 453 874	10·0
11·0	7288·489 339 821 248 106 179	6948·858 659 812 163 230 818	0·000 006 243 020 547 653	0·000 006 520 860 674 582	11·0

Table IV.

x.	Log *x* − E − 1.	*x*.
0·1	3·418 516 608 652 458 132 828 711 486 060	0·1
0·2	2·725 369 428 092 512 823 411 479 364 602	0·2
0·3	2·319 904 319 984 348 441 433 466 249 138	0·3
0·4	2·032 222 247 532 567 513 994 247 243 144	0·4
0·5	1·809 078 696 218 357 758 227 952 152 834	**0·5**
0·6	1·626 757 139 424 403 132 016 234 127 680	0·6
0·7	1·472 606 459 597 144 827 723 358 742 617	0·7
0·8	1·339 075 066 972 622 204 577 015 121 686	0·8
0·9	1·221 292 031 316 238 750 038 221 012 216	0·9
1·0	1·115 931 515 658 412 448 810 720 031 376	**1·0**
1·1	1·020 621 335 854 087 588 766 767 908 095	1·1
1·2	0·933 609 958 864 457 822 599 002 006 222	1·2
1·3	0·853 567 251 190 921 396 775 224 044 495	1·3
1·4	0·779 459 279 037 199 518 306 126 621 159	1·4
1·5	0·710 466 407 550 248 066 832 706 915 912	**1·5**
1·6	0·645 927 886 412 676 895 159 783 000 228	1·6
1·7	0·585 303 264 596 242 052 579 176 868 188	1·7
1·8	0·528 144 850 756 293 440 620 988 890 758	1·8
1·9	0·474 077 629 486 017 672 819 684 054 173	1·9
2·0	0·422 784 335 098 467 139 393 487 909 918	**2·0**
2·1	0·373 994 170 929 035 136 328 113 505 694	2·1
2·2	0·327 474 155 294 142 279 349 535 786 637	2·2
2·3	0·283 022 392 723 308 442 021 958 654 250	2·3
2·4	0·240 462 778 304 512 513 181 769 884 763	2·4
2·5	0·199 640 783 784 257 383 627 192 819 608	**2·5**
2·6	0·160 420 070 630 976 087 357 991 923 037	2·6
2·7	0·122 679 742 648 129 058 642 975 775 293	2·7
2·8	0·086 312 098 477 254 208 888 894 499 701	2·8
2·9	0·051 220 778 665 984 105 645 439 453 698	2·9
3·0	0·017 319 226 990 302 757 415 474 794 454	**3·0**
3·1	0·015 470 595 832 688 113 100 452 838 482	3·1
3·2	0·047 219 294 147 268 414 257 449 121 230	3·2
3·3	0·077 990 952 814 022 102 628 477 328 827	3·3
3·4	0·107 843 915 963 703 256 838 055 253 271	3·4
3·5	0·136 831 452 836 955 546 877 400 590 609	**3·5**
3·6	0·165 002 329 803 651 868 796 243 230 701	3·6
3·7	0·192 401 303 991 766 311 539 384 184 971	3·7
3·8	0·219 069 551 073 927 636 597 548 067 286	3·8
3·9	0·245 045 037 477 188 294 620 021 192 428	3·9
4·0	0·270 362 845 461 478 170 023 744 211 540	**4·0**
4·1	0·295 055 458 051 849 671 038 051 886 977	4·1
4·2	0·319 153 009 630 910 173 089 118 615 764	4·2
4·3	0·342 683 507 041 104 290 644 131 027 286	4·3
4·4	0·365 673 025 265 803 030 067 696 334 821	4·4
4·5	0·388 145 881 117 861 624 562 538 321 011	**4·5**
4·6	0·410 124 787 836 636 867 395 273 467 208	4·6
4·7	0·431 630 993 057 600 453 992 239 183 712	4·7
4·8	0·452 684 402 255 432 796 235 462 236 695	4·8
4·9	0·473 303 689 458 168 477 381 994 000 826	4·9
5·0	0·493 506 396 775 687 925 790 039 301 850	**5·0**
5·1	0·513 309 024 071 867 638 816 068 368 736	5·1
5·2	0·532 727 109 928 969 222 059 240 198 422	5·2
5·3	0·551 775 304 899 663 701 315 757 652 969	5·3
5·4	0·570 467 437 911 816 250 774 256 346 166	5·4
5·5	0·588 816 576 580 012 785 833 991 425 131	**5·5**
5·6	0·606 835 082 082 691 100 528 337 621 758	5·6
5·7	0·624 534 659 182 092 018 575 561 182 750	5·7
5·8	0·641 926 401 893 961 203 771 792 667 760	5·8
5·9	0·659 020 835 253 261 317 787 338 887 660	5·9
6·0	0·675 827 953 569 642 552 001 757 327 004	**6·0**

Old numeral type (518) in Tables IV and VI denotes negative quantities.

Table V.

n.	$S_n - 1$.
6	1·450
7	1·592 857 142 857 142
8	1·717 857 142 857 142
9	1·828 968 253 968 253
10	1·928 968 253 968 253
11	2·019 877 344 877 344
12	2·103 210 678 210 678
13	2·180 133 755 133 755
14	2·251 562 326 562 326
15	2·318 228 993 228 993
16	2·380 728 993 228 993 228
17	2·439 552 522 639 757 934 875 580
18	2·495 108 078 195 313 490 431 135
19	2·547 739 657 142 681 911 483 766
20	2·597 739 657 142 681 911 483 766
21	2·645 358 704 761 729 530 531 385
22	2·690 813 250 216 274 985 076 839
23	2·734 291 511 085 840 202 468 143
24	2·775 958 177 752 506 869 134 809
25	2·815 958 177 752 506 869 134 809
26	2·854 419 716 314 045 326
27	2·891 456 753 351 082 363
28	2·927 171 039 065 368 077

Table VI.

n.	Log m_n.	Log μ_n.	n.
1	$\bar{1}$·096 9100 130	$\bar{1}$·574 0312 677	1
2	$\bar{1}$·750 1225 267	$\bar{1}$·494 8500 216	2
3	0·017 7287 670	$\bar{1}$·942 0080 530	3
4	0·185 0461 017	0·148 0625 355	4
5	0·306 4250 276	0·284 4307 339	5
6	0·401 5441 329	0·386 9446 243	6
7	0·479 6986 776	0·469 2959 172	7
8	0·546 0025 441	0·538 2122 997	8
9	0·603 5653 464	0·597 5123 636	9
10	0·654 4172 149	0·649 5782 291	10
11	0·699 9559 173	0·695 9987 648	11
12	0·741 1844 390	0·737 8880 704	12
13	0·778 8466 780	0·776 0582 609	13
14	0·813 5095 056	0·811 1199 839	14
15	0·845 6147 498	0·843 5442 120	15
16	0·875 5134 181	0·873 7019 682	16
17	0·903 4889 714	0·901 8908 298	17
18	0·929 7735 966	0·928 3531 718	18
19	0·954 5598 602	0·953 2890 635	19
20	0·978 0092 314	0·976 8655 981	20
21	1·000 2584 317	0·999 2237 809	21
22	1·021 4242 434	1·020 4837 027	22
23	1·041 6072 046	1·040 7484 905	23
24	1·060 8944 872	1·060 1073 651	24
25	1·079 3621 644	1·078 6380 383	25